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Magneto-Convection of Immiscible Fluids in a Vertical Channel Using Robin Boundary Conditions

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ABSTRACT
The effects of viscous dissipation on fully developed two fluid magnetohydrodynamic flow in the presence of constant electric field in a vertical channel is investigated using Robin boundary conditions. The fluids in both the regions are incompressible, electrically conducting and the transport properties are assumed to be constant. The plate exchanges heat with an external fluid. Both conditions of equal and different reference temperatures of the external fluid are considered. First, the simple cases of the negligible Brinkman number or the negligible Grashof number are solved analytically. Then, the combined effects of buoyancy forces and viscous dissipation are analyzed by a perturbation series method valid for small values of perturbation parameter. To relax the condition on the perturbation parameter, the flow fields are solved by using the differential transform method. The results are presented graphically for different values of the mixed convection parameter, Hartman number, perturbation parameter, viscosity ratio, width ratio, conductivity ratio and Biot numbers for both open and short circuit. The effects of these parameters on the Nusselt number at the walls is also drawn. It is found that the solutions obtained by perturbation method and differential transform method agree very well for small values of perturbation parameter.

I. INTRODUCTION:

The interaction between the conducting fluid and the magnetic field radically modifies the flow, with attendant effects on such important flow properties as pressure drop and heat transfer, the detailed nature of which is strongly dependent on the orientation of the magnetic field. The advent of technology that involves the MHD power generators, MHD devices, nuclear engineering and the possibility of thermonuclear power has created a great practical need for understanding the dynamics of conducting fluids. Moreover, there has been interest in studying the flow of electrically conducting fluids over surfaces. On the other hand, the heat transfer rates can be controlled using a magnetic field. One of the ways of studying magnetohydrodynamic heat transfer field is the electromagnetic field, which is used to control the heat transfer as in the convection flows and aerodynamic heating.

The use of electrically conducting fluids under the influence of magnetic fields in various industries has lead to a renewed interest in investigating hydrodynamic flow and heat transfer in different geometries. For example, Sparrow and Cess [1] and Umavathi [2] studied magneto convection in vertical channel in the presence of electric field. Bhargava et al. [3] have studied the effect of magnetic field on the free convection flow of a micropolar fluid between two parallel porous vertical plates. Hayat et al. [4] have studied the Hall effects on the unsteady hydrodynamic oscillatory flow of a second grade fluid. Recently, Umavathi et al. [5] numerically studied fully developed magneto convection flow in a vertical rectangular duct.

There has been some theoretical and experimental work on stratified laminar flow of two immiscible liquids in the horizontal pipe (Charles and Redburger [6], Pacham and Shail [7]). The interest in this configuration stems from the possibility of reducing the power required to pump oil in a pipeline by suitable addition of water. Shail [8] investigated theoretically the possibility of using a two-fluid system to obtain increased flow rates in an electromagnetic pump. Specially, Shail [8] studied Hartmann flow of a conducting fluid which is pumped electromagnetically in a channel bounded by two parallel horizontal insulating plates of infinite extent, there being a layer of non conducting fluid between the conducting liquid and the upper channel wall.

Another physical phenomenon is the case in which the two immiscible conducting fluids flow past permeable beds. Recent advances in two-fluid flow researchers are remarkable and many things have been clarified about various phenomena in two-fluid flow. However, of course, there are still many more things to be studied in order to achieve sufficient understanding and satisfactory prediction about two-fluid flow. In particular, the microscopic structures of two-fluid flow, such as velocity and phase distributions, interfacial structures and turbulence phenomena, are quite important topics and much effort has been made in these research areas in recent years. This is partly due to scientific interest in the physical phenomenon of two fluid flow and partly due to industrial demands for more precise predictions of two fluid flow behavior in various industrial devices. The coal-fired magnetohydrodynamic generator channel is subjected to an unusual severe thermal environment. Postlethwaite and Sluyter [9] presents an overview of the heat transfer problems associated with a MHD generator.

Malashetty and Leela [10] reported closed form solutions for the two fluid flow and heat transfer situations in a horizontal channel for which both phases are electrically conducting. Malashetty and Umavathi [11] studied two fluid MHD flow and heat transfer in an inclined channel in the presence of buoyancy effects for the situations where only one of the phases is electrically conducting. Malashetty et al. [12-14] analyzed the problem of fully developed two fluid magnetohydrodynamic flows with and without applied electric field in an inclined channel. Umavathi et al. [15-17] studied steady and unsteady magnetohydrodynamic two fluid flow in a vertical and horizontal channel. Prathap Kumar et al. [18, 19] also studied mixed convection of magnetohydrodynamic two fluid flow in a vertical enclosure.

The developing flow with asymmetric wall temperature has been considered by Ingham et al. [20], with particular reference to situations where reverse flow occurs. On the other hand, Barletta [21] and Zanchini [22] have pointed out that relevant effects of viscous dissipation on the temperature profiles and on the Nusselt numbers may occur in the fully developed laminar forced convection in tubes. Thus, an analysis of the effect of viscous dissipation in the fully developed mixed convection in vertical ducts appears as interesting. Several studies on mixed convection problems for a Newtonian fluid in a vertical channel have already been presented in literature. In particular, some analytical solutions for the fully developed flow have been performed. The boundary conditions of uniform wall temperatures have been analyzed by Aung and Worku [23]. The boundary conditions of uniform wall temperatures on a wall and a uniform wall heat fluxes, have been studied by Cheng et al. [24]. The effect of viscous dissipation on the velocity and on the temperature fields have been analyzed by Barletta [25] for the boundary conditions of uniform wall temperatures and by Zanchini [26] for boundary conditions of third kind.

In the past, the laminar forced convection heat transfer in the hydrodynamic entrance region of a flat rectangular channel wall has been investigated either for the temperature boundary conditions of the first kind, characterized by prescribed wall temperature (Stephan [27], Hwang and Fan[28]), or boundary conditions of second kind, expressed by prescribed wall temperature heat flux (Siegal and Sparrow[29]). A more realistic condition in many applications, however, will be the temperature boundary conditions of third kind: the local wall heat flux is a linear function of the local wall temperature.

The differential transform scheme (DTM) is a method for solving a wide range of problems whose mathematical models yield equations or systems of equations involving algebraic, differential, integral and integro-differential equations (Arikhoglu and Ozkol [30] and Biazar et al. [31]). The concept of the differential transform was first proposed by Zhou [32], and its main applications therein is solved for both linear and non-linear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of polynomials. It is different from the high-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. With this technique, the given differential equation and related initial and boundary conditions are transformed into a recurrence equation that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. Therefore the differential transform method can overcome the restrictions and limitations of perturbation techniques so that it provides us with a possibility to analyze strongly nonlinear problems. In recent years the application of differential transform theory has been appeared in many researches (Rashidi et al. [33], and Ganji et al. [34]).

Recently, Umavathi and Santhosh [35, 36], Umavathi and Jaweria [37] studied mixed convection in vertical channel using boundary conditions of third kind. The aim of this paper is to extend the analysis performed by Zanchini [26] for electrically conducting immiscible fluids.

II. MATHEMATICAL FORMULATION:

The geometry under consideration illustrated in Figure 1 consists of two infinite parallel plates maintained at equal or different constant temperatures extending in the X and Z directions. The region $-h_1/2 \le Y \le 0$ is occupied by electrically conducting fluid of density ρ_1, viscosity μ_1, thermal conductivity k_1, and thermal expansion coefficient β_1, and the region $0 \le Y \le h_2/2$ is occupied by another immiscible electrically conducting fluid of density ρ_2, viscosity μ_2, thermal conductivity k_2, and thermal expansion coefficient β_2.

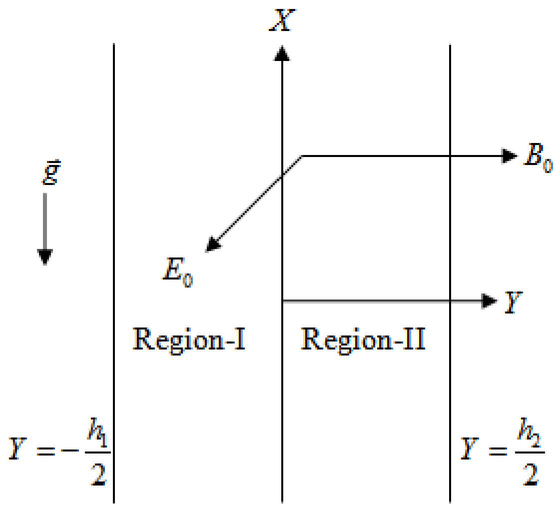

Figure 1 Physical configuration

The fluids are assumed to have constant properties except the density in the buoyancy term in the momentum equation $\rho_1 = \rho_0\left[1-\beta_1\left(T_1-T_0\right)\right]$ and $\rho_2 = \rho_0\left[1-\beta_2\left(T_2-T_0\right)\right]$. A constant magnetic field of strength B_0 is applied normal to the plates and a uniform electric field E_0 is applied perpendicular to the plates. A fluid rises in the channel driven by buoyancy forces. The transport properties of both fluids are assumed to be constant. We consider the fluids to be incompressible and the flow is steady, laminar, and fully developed. It is assumed that the only non-zero component of the velocity $\vec{q}$ is the *X*-component $U_i\,(i=1,2)$. Thus, as a consequence of the mass balance equation, one obtains

$$\frac{\partial U_i}{\partial X}=0 \tag{1}$$

so that U_i depends only on Y. The stream wise and the transverse momentum balance equations yields

Region-I

$$g\beta_1\left(T_1-T_0\right)-\frac{1}{\rho_1}\frac{\partial P}{\partial X}+\nu_1\frac{d^2U_1}{dY^2}-\frac{\sigma_{e1}}{\rho_1}\left(E_0+B_0U_1\right)B_0=0 \tag{2}$$

Region-II

$$g\beta_2\left(T_2-T_0\right)-\frac{1}{\rho_2}\frac{\partial P}{\partial X}+\nu_2\frac{d^2U_1}{dY^2}-\frac{\sigma_{e2}}{\rho_2}\left(E_0+B_0U_2\right)B_0=0 \tag{3}$$

and Y-momentum balance equation in both the regions can be expressed as

$$\frac{\partial P}{\partial Y}=0 \tag{4}$$

where $P = p+\rho_0 g x$ (assuming $p_1 = p_2 = p$) is the difference between the pressure and hydrostatic pressure. On account of equation (4), P depends only on X so that equations (2) and (3) can be rewritten as

Region-I

$$T_1-T_0=\frac{1}{g\beta_1\rho_1}\frac{dP}{dX}-\frac{\nu_1}{g\beta_1}\frac{d^2U_1}{dY^2}+\frac{\sigma_{e1}}{g\beta_1\rho_1}\left(E_0+B_0U_1\right)B_0 \tag{5}$$

Region-II

$$T_2-T_0=\frac{1}{g\beta_2\rho_2}\frac{dP}{dX}-\frac{\nu_2}{g\beta_2}\frac{d^2U_2}{dY^2}+\frac{\sigma_{e2}}{g\beta_2\rho_2}\left(E_0+B_0U_2\right)B_0 \tag{6}$$

From equations (5) and (6) one obtains

Region-I

$$\frac{\partial T_1}{\partial X} = \frac{1}{g\beta_1\rho_1}\frac{d^2P}{dX^2} \tag{7}$$

$$\frac{\partial T_1}{\partial Y} = -\frac{\nu_1}{g\beta_1}\frac{d^3U_1}{dY^3} + \frac{\sigma_{e1}}{g\beta_1\rho_1}B_0^2\frac{dU_1}{dY} \tag{8}$$

$$\frac{\partial^2 T_1}{\partial Y^2} = -\frac{\nu_1}{g\beta_1}\frac{d^4U_1}{dY^4} + \frac{\sigma_{e1}}{g\beta_1\rho_1}B_0^2\frac{d^2U_1}{dY^2} \tag{9}$$

Region-II

$$\frac{\partial T_2}{\partial X} = \frac{1}{g\beta_2\rho_2}\frac{d^2P}{dX^2} \tag{10}$$

$$\frac{\partial T_2}{\partial Y} = -\frac{\nu_2}{g\beta_2}\frac{d^3U_2}{dY^3} + \frac{\sigma_{e2}}{g\beta_2\rho_2}B_0^2\frac{dU_2}{dY} \tag{11}$$

$$\frac{\partial^2 T_2}{\partial Y^2} = -\frac{\nu_2}{g\beta_2}\frac{d^4U_2}{dY^4} + \frac{\sigma_{e2}}{g\beta_2\rho_2}B_0^2\frac{d^2U_2}{dY^2} \tag{12}$$

Both the walls of the channel will be assumed to have a negligible thickness and to exchange heat by convection with an external fluid. In particular, at $Y = -h_1/2$ the external convection coefficient will be considered as uniform with the value q_1 and the fluid in the region $-h_1/2 \le Y \le 0$ will be assumed to have a uniform reference temperature T_{q_1}. At $Y = h_2/2$ the external convection coefficient will be considered as uniform with the value q_2 and the fluid in the region $0 \le Y \le h_2/2$ will be supposed to have a uniform reference temperature $T_{q_2} \ge T_{q_1}$. Therefore, the boundary conditions on the temperature field can be expressed as

$$-k_1\left.\frac{\partial T_1}{\partial Y}\right|_{Y=-\frac{h_1}{2}} = q_1\left[T_{q_1} - T_1\left(X, -h_1/2\right)\right] \tag{13}$$

$$-k_2\left.\frac{\partial T_2}{\partial Y}\right|_{Y=\frac{h_2}{2}} = q_2\left[T_2\left(X, h_2/2\right) - T_{q_2}\right] \tag{14}$$

On account of equations (8) and (11), equations (13) and (14) can be rewritten as

$$\frac{d^3U_1}{dY^3} - \frac{\sigma_{e1}B_0^2}{\mu_1}\frac{dU_1}{dY} = \frac{g\beta_1}{k_1\nu_1}q_1\left[T_{q_1} - T_1\left(X, -h_1/2\right)\right] \quad \text{at } Y = -\frac{h_1}{2} \tag{15}$$

$$\frac{d^3U_2}{dY^3} - \frac{\sigma_{e2}B_0^2}{\mu_2}\frac{dU_2}{dY} = \frac{g\beta_2}{k_2\nu_2}q_2\left[T_2\left(X, h_2/2\right) - T_{q_2}\right] \quad \text{at } Y = \frac{h_2}{2} \tag{16}$$

On account of Equations (5) and (6), there exist a constant A such that

$$\frac{dP}{dX} = A \tag{17}$$

For the problem under examination, the energy balance equation in the presence of viscous dissipation can be written as

Region-I

$$\frac{d^2T_1}{\partial Y^2} = -\frac{\nu_1}{\alpha_1 c_p}\left(\frac{dU_1}{dY}\right)^2 - \frac{\sigma_{e1}}{\rho_1 c_p \alpha_1}\left(E_0 + B_0 U_1\right)^2 \tag{18}$$

Region-II

$$\frac{d^2T_2}{\partial Y^2}=-\frac{\nu_2}{\alpha_2 c_p}\left(\frac{dU_2}{dY}\right)^2-\frac{\sigma_{e2}}{\rho_2 c_p\alpha_2}\left(E_0+B_0U_2\right)^2 \tag{19}$$

From equations (9), (18), (12) and (19) allow one to obtain differential equations for U_i namely

Region-I

$$\frac{d^4U_1}{dY^4}=\frac{g\beta_1}{\alpha_1 c_p}\left(\left(\frac{dU_1}{dY}\right)^2+\frac{\sigma_{e1}}{\mu_1}\left(E_0+B_0U_1\right)^2\right)+\frac{\sigma_{e1}B_0^2}{\mu_1}\frac{d^2U_1}{dY^2} \tag{20}$$

Region-II

$$\frac{d^4U_2}{dY^4}=\frac{g\beta_2}{\alpha_2 c_p}\left(\left(\frac{dU_2}{dY}\right)^2+\frac{\sigma_{e2}}{\mu_2}\left(E_0+B_0U_2\right)^2\right)+\frac{\sigma_{e2}B_0^2}{\mu_2}\frac{d^2U_2}{dY^2} \tag{21}$$

The boundary conditions on U_i are

$$U_1\left(-h_1/2\right)=U_2\left(h_2/2\right)=0 \tag{22}$$

together with equations (15) and (16) which on account of equations (5) and (6) can be rewritten as

$$\left(\frac{d^3U_1}{dY^3}-\frac{\sigma_{e1}B_0^2}{\mu_1}\frac{dU_1}{dY}-\frac{q_1}{k_1}\frac{d^2U_1}{dY^2}\right)_{y=-h_1/2}=\frac{q_1}{k_1}\frac{g\beta_1}{\nu_1}\left(T_{q_1}-T_0\right)+\frac{q_1}{k_1}\left(-\frac{A}{\mu_1}-\frac{\sigma_{e1}}{\mu_1}\left(E_0+B_0U_1\right)B_0\right)$$

$$\left(\frac{d^3U_2}{dY^3}-\frac{\sigma_{e2}B_0^2}{\mu_2}\frac{dU_2}{dY}+\frac{q_2}{k_2}\frac{d^2U_2}{dY^2}\right)_{y=h_2/2}=\frac{q_2}{k_2}\frac{g\beta_2}{\nu_2}\left(T_0-T_{q_2}\right)+\frac{q_2}{k_2}\left(-\frac{A}{\mu_2}+\frac{\sigma_{e2}}{\mu_2}\left(E_0+B_0U_2\right)B_0\right) \tag{23}$$

The continuity of velocity, shear stress, and temperature and heat flux is assumed to be continuous at the interface as follows

$$U_1(0)=U_2(0)\ ,\ \mu_1\frac{dU_1(0)}{dy}=\mu_2\frac{dU_2(0)}{dy}\ ,T_1(0)=T_2(0)\ ,\ k_1\frac{dT_1(0)}{dy}=k_2\frac{dT_2(0)}{dy} \tag{24}$$

Equations (20)-(24) determine the velocity distribution. They can be written in a dimensionless form by means of the following dimensionless parameters

$$u_1=\frac{U_1}{U_0^{(1)}}\ ,\ u_2=\frac{U_2}{U_0^{(2)}},\ y_1=\frac{Y_1}{D_1},\ y_2=\frac{Y_2}{D_2}\ ,\ Gr=\frac{g\beta_1\Delta TD_1^3}{\nu_1^2},\ \mathrm{Re}=\frac{U_0^{(1)}D_1}{\nu_1},\ \ Br=\frac{U_0^{(1)^2}\mu_1}{k_1\Delta T}$$

$$\Lambda=\frac{Gr}{\mathrm{Re}},\ \ \theta_1=\frac{T_1-T_0}{\Delta T},\ \theta_2=\frac{T_2-T_0}{\Delta T},\ R_T=\frac{T_2-T_1}{\Delta T},\ M^2=\frac{\sigma_{e1}B_0^2D_1^2}{\mu_1},\ E=\frac{E_0}{B_0U_0^{(1)}} \tag{25}$$

where $D=2h$ is the hydraulic diameter. The reference velocity and the reference temperature are given by

$$U_0^{(1)}=-\frac{AD_1^2}{48\mu_1},\ U_0^{(2)}=-\frac{AD_2^2}{48\mu_2},\ T_0=\frac{T_{q_1}+T_{q_2}}{2}+s\left(\frac{1}{Bi_1}-\frac{1}{Bi_2}\right)\left(T_{q_2}-T_{q_1}\right) \tag{26}$$

Moreover, the temperature difference ΔT is given by $\Delta T=T_{q_2}-T_{q_1}$ if $T_{q_1}<T_{q_2}$. As a consequence, the dimensionless parameter R_T can only take the values 0 or 1. More precisely, the temperature difference ratio R_T is equal to 1 for asymmetric heating i.e. $T_{q_1}<T_{q_2}$, while R_T =0 for symmetric heating i.e. $T_{q_1}=T_{q_2}$, respectively. Equation (17) implies that A can be either positive or negative. If $A<0$, then U_0^i, Re and Λ are negative, i.e. the flow is downward. On the other hand, if $A>0$, the flow is upward, so that U_0^i, Re, and Λ are positive. Using equations (25) and (26), equations (20)-(24) becomes

Region-I

$$\frac{d^4u_1}{dy^4}-M^2\frac{d^2u_1}{dy^2}=\Lambda Br\left(\left(\frac{du_1}{dy}\right)^2+M^2E^2+M^2u_1^2+2M^2Eu_1\right) \tag{27}$$

Region-II

$$\frac{d^4u_2}{dy^4}-M^2h^2\sigma r\, m\frac{d^2u_1}{dy^2}= \Lambda\, Br\, bnkh^2\left(mh^2\left(\frac{du_2}{dy}\right)^2+M^2\sigma r\left(E^2+m^2h^4u_2^2+2mh^2\,E\,u_2\right)\right) \tag{28}$$

The boundary and interface conditions becomes

$$u_1(-1/4)=u_2(1/4)=0,\quad u_1(0)=mh^2u_2(0),\quad \frac{du_1(0)}{dy}=h\frac{du_2(0)}{dy},$$

$$\left(\frac{d^2u_1}{dy^2}-M^2u_1\right)=\frac{1}{nb}\frac{d^2u_2}{dy^2}-\frac{M^2\,\sigma r\, mh^2}{nb}u_2-48+M^2E+\frac{48}{nb}-\frac{M^2E\,\sigma r}{nb}\quad \text{at}\quad y=0,$$

$$\left(\frac{d^3u_1}{dy^3}-M^2\frac{du_1}{dy}\right)=\frac{1}{nbkh}\frac{d^3u_2}{dy^3}-\frac{M^2\,\sigma r\, mh}{nbk}\frac{du_2}{dy}\quad \text{at}\quad y=0,$$

$$\left(\frac{d^2u_1}{dy^2}+\frac{M^2}{Bi_1}\frac{du_1}{dy}-\frac{1}{Bi_1}\frac{d^3u_1}{dy^3}-M^2u_1\right)_{y=-1/4}=-48+\frac{R_T}{2}\Lambda s\left(1+\frac{4}{Bi_1}\right)+M^2E,$$

$$\left(\frac{d^2u_2}{dy^2}+\frac{1}{Bi_2}\frac{d^3u_2}{dy^3}-\frac{M^2h^2\sigma r\, m}{Bi_2}\frac{du_2}{dy}-M^2h^2\sigma r\, mu_2\right)_{y=1/4}= -48-\frac{R_T}{2}sbn\Lambda\left(1+\frac{4}{Bi_2}\right)+M^2\sigma r\;E \tag{29}$$

Basic Idea of Differential Transformation Method (DTM)

The transformation of the k[th] derivative of a function in one variable is as follows:

$$\bar{U}(k)=\frac{1}{k!}\left[\frac{d^ku(y)}{dy^k}\right]_{y=0} \tag{30}$$

where $u(y)$ is the original function and $\bar{U}(k)$ is the transformed function which is called the T-function. The differential inverse transform of $\bar{U}(k)$ is defined as

$$u(y)=\sum_{k=0}^{\infty}\bar{U}(k)\,y^k \tag{31}$$

Equation (31) implies that the concept of the differential transformation is derived from Taylor's series expansion (see Zhou, [32]), but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by iterative procedure that is described by the transformed equations of the original functions. In real applications, the function $u(y)$ is a finite series and hence equation (31) can be written as

$$u(y)=\sum_{k=0}^{n}\bar{U}(k)\,y^k \tag{32}$$

and equation (31) implies that $u(y)=\sum_{k=n+1}^{\infty}\bar{U}(k)\,y^k$ is neglected as it is small. Usually, the values of *n* are decided by a convergence of the series coefficients. Mathematical operations performed by differential transform method are listed in Table 1.

III. SOLUTIONS

Case-I

The solution of equations (27) and (28) using boundary and interface conditions in equation (29) in the absence of viscous dissipation term ($Br = 0$) is given by

Region-I

$$u_1 = c_1 + c_2 y + c_3 Cosh(My) + c_4 Sinh(My) \tag{33}$$

Region-II

$$u_2 = c_5 + c_6 y + c_7 Cosh(p_1 y) + c_8 Sinh(p_1 y) \tag{34}$$

where $p_1 = \sqrt{M^2 h^2 \sigma r\, m}$, and using equation (29) in equations (5) and (6), the energy balance equations becomes

Region-I

$$\theta_1 = \frac{1}{\Lambda}\left(-48 - \frac{d^2 u_1}{dy^2} + M^2 u_1 + M^2 E \right) \tag{35}$$

Region-II

$$\theta_2 = \frac{1}{\Lambda bn}\left(-48 - \frac{d^2 u_2}{dy^2} + mh^2 M^2 \sigma r\, u_2 + M^2 E \sigma r \right) \tag{36}$$

Using the expressions obtained in equations (33) and (34) the energy balance equations (35) and (36) becomes

Region-I

$$\theta_1 = -\frac{1}{\Lambda}\left(48 - M^2 (c_1 + c_2 y) + M^2 E \right) \tag{37}$$

Region-II

$$\theta_2 = \frac{1}{\Lambda nb}\left(48 - M^2 h^2 m \sigma r (c_5 + c_6 y) + M^2 E \sigma r \right) \tag{38}$$

Case-II

The solution of equation (27) and (28) can be obtained when buoyancy forces are negligible ($\Lambda = 0$) and viscous dissipation is dominating ($Br \neq 0$), so that purely forced convection occurs. For this case, solutions of equations (27) and (28), using the boundary and interface conditions given by equation (29), the velocities are given by

Region-I

$$u_1 = l_1 + l_2 y + l_3 Cosh(My) + l_4 Sinh(My) \tag{39}$$

Region-II

$$u_2 = l_5 + l_6 y + l_7 Cosh(p_1 y) + l_8 Sinh(p_1 y) \tag{40}$$

The energy balance equations (18) and (19) in non-dimensional form can also be written as

Region-I

$$\frac{d^2\theta_1}{dy^2} = -Br\left(\left(\frac{du_1}{dy} \right)^2 + M^2 u_1^2 + M^2 E^2 + 2M^2 E u_1 \right) \tag{41}$$

Region-II

$$\frac{d^2\theta_2}{dy^2} = -Br\left(mkh^4 \left(\frac{du_2}{dy} \right)^2 + M^2 \sigma r k h^2 \left(E^2 + m^2 h^4 u_2^2 + 2Eh^2 m u_2 \right) \right) \tag{42}$$

The boundary and interface conditions for temperature are

$$\left.\frac{d\theta_1}{dy}\right|_{y=-1/4} - Bi_1 \theta_1(-1/4) = \frac{Bi_1 R_T s}{2}\left(1 + \frac{4}{Bi_1} \right), \left.\frac{d\theta_2}{dy}\right|_{y=1/4} + Bi_2 \theta_2(1/4) = \frac{Bi_2 R_T s}{2}\left(1 + \frac{4}{Bi_2} \right),$$

$$\theta_1(0)=\theta_2(0),\quad \frac{d\theta_1(0)}{dy}=\frac{1}{kh}\frac{d\theta_2(0)}{dy} \tag{43}$$

Using equations (39) and (40) , solving equations (41) and (42) we obtain

Region-I

$$\begin{aligned}\theta_1=-Br\Big(&G_1Cosh(2My)+G_2Sinh(2My)+G_3Cosh(My)+G_4Sinh(My)+\\&G_5yCosh(My)+G_6ySinh(My)+G_7y^4+G_8y^3+G_9y^2\Big)+d_1y+d_2\end{aligned} \tag{44}$$

Region-II

$$\begin{aligned}\theta_2=-Br\Big(&G_{10}Cosh(2p_1y)+G_{11}Sinh(2p_1y)+G_{12}Cosh(p_1y)+G_{13}Sinh(p_1y)+\\&G_{14}yCosh(p_1y)+G_{15}ySinh(p_1y)+G_{16}y^4+G_{17}y^3+G_{18}y^2\Big)+d_3y+d_4\end{aligned} \tag{44}$$

Perturbation Method (PM):

We solve equations (27) and (28) using the perturbation method with a dimensionless parameter $|\varepsilon|$ (<<1) defined as

$$\varepsilon=\Lambda Br \tag{46}$$

and does not depend on the reference temperature difference ΔT. To this end the solutions are assumed in the form

$$u(y)=u_0(y)+\varepsilon u_1(y)+\varepsilon^2u_2(y)+...=\sum_{n=0}^{\infty}\varepsilon^n u_n(y) \tag{47}$$

Substituting equation (47) in equation (27) and (28) and equating the coefficients of like powers of ε to zero, we obtain the zero and first order equations as follows:

Region-I

Zero-order equations

$$\frac{d^4u_{10}}{dy^4}-M^2\frac{d^2u_{10}}{dy^2}=0 \tag{48}$$

First-order equations

$$\frac{d^4u_{11}}{dy^4}-M^2\frac{d^2u_{11}}{dy^2}=\left(\frac{du_{10}}{dy}\right)^2+M^2E^2+M^2u_{10}^2+2M^2Eu_{10} \tag{49}$$

Region-II

Zero-order equations

$$\frac{d^4u_{20}}{dy^4}-M^2h^2\sigma r\,m\frac{d^2u_2}{dy^2}=0 \tag{50}$$

First-order equations

$$\begin{aligned}&\frac{d^4u_{21}}{dy^4}-M^2h^2\sigma r\,m\frac{d^2u_{21}}{dy^2}=\\&\quad\Lambda\,Br\,bnkh^2\left(mh^2\left(\frac{du_{20}}{dy}\right)^2+M^2\sigma r\left(E^2+m^2h^4u_{20}^2+2mh^2\,E\,u_{20}\right)\right)\end{aligned} \tag{51}$$

The corresponding boundary and interface conditions given by equation (29) for the zeroth and first order reduces to

Zeroth-order

$$u_{10}(-1/4)=u_{20}(1/4)=0,\ u_{10}(0)=mh^2u_{20}(0),\quad \frac{du_{10}(0)}{dy}=h\frac{du_{20}(0)}{dy},$$

$$\left(\frac{d^2u_{10}}{dy^2}-M^2u_{10}\right)=\frac{1}{nb}\frac{d^2u_{20}}{dy^2}-\frac{M^2\,\sigma r\,mh^2}{nb}u_{20}-48+M^2E+\frac{48}{nb}-\frac{M^2E\,\sigma r}{nb}\quad \text{at}\quad y=0,$$

$$\left(\frac{d^3u_{10}}{dy^3}-M^2\frac{du_{10}}{dy}\right)=\frac{1}{nbkh}\frac{d^3u_{20}}{dy^3}-\frac{M^2\,\sigma r\,mh}{nbk}\frac{du_{20}}{dy}\quad \text{at } y=0,$$

$$\left(\frac{d^2u_{10}}{dy^2}+\frac{M^2}{Bi_1}\frac{du_{10}}{dy}-\frac{1}{Bi_1}\frac{d^3u_{10}}{dy^3}-M^2u_{10}\right)_{y=-1/4}=-48+\frac{R_T}{2}\Lambda s\left(1+\frac{4}{Bi_1}\right)+M^2E,$$

$$\left(\frac{d^2u_{20}}{dy^2}+\frac{1}{Bi_2}\frac{d^3u_{20}}{dy^3}-\frac{M^2h^2\sigma r\,m}{Bi_2}\frac{du_{20}}{dy}-M^2h^2\sigma r\,mu_{20}\right)_{y=1/4}=-48 \qquad (52)$$

$$-\frac{R_T}{2}sbn\Lambda\left(1+\frac{4}{Bi_2}\right)+M^2\sigma r\,E$$

First-order

$$u_{11}(-1/4)=u_{21}(1/4)=0,\; u_{11}(0)=mh^2u_{21}(0),\; \frac{du_{11}(0)}{dy}=h\frac{du_{21}(0)}{dy},$$

$$\left(\frac{d^2u_{11}}{dy^2}-M^2u_{11}\right)=\frac{1}{nb}\frac{d^2u_{21}}{dy^2}-\frac{M^2\,\sigma r\,mh^2}{nb}u_{21}\quad \text{at } y=0,$$

$$\left(\frac{d^3u_{11}}{dy^3}-M^2\frac{du_{11}}{dy}\right)=\frac{1}{nbkh}\frac{d^3u_{21}}{dy^3}-\frac{M^2\,\sigma r\,mh}{nbk}\frac{du_{21}}{dy}\quad \text{at } y=0,$$

$$\left(\frac{d^2u_{11}}{dy^2}+\frac{M^2}{Bi_1}\frac{du_{11}}{dy}-\frac{1}{Bi_1}\frac{d^3u_{11}}{dy^3}-M^2u_{11}\right)_{y=-1/4}=0,$$

$$\left(\frac{d^2u_{21}}{dy^2}+\frac{1}{Bi_2}\frac{d^3u_{21}}{dy^3}-\frac{M^2h^2\sigma r\,m}{Bi_2}\frac{du_{21}}{dy}-M^2h^2\sigma r\,mu_{21}\right)_{y=1/4}=0 \qquad (53)$$

Solutions of zeroth-order equations (48) and (50) using boundary and interface conditions (52) are

$$u_{10}=z_1+z_2y+z_3Cosh(My)+z_4Sinh(My) \qquad (54)$$

$$u_{20}=z_5+z_6y+z_7Cosh(p_1y)+z_8Sinh(p_1y) \qquad (55)$$

Solutions of first-order equations (49) and (51) using boundary and interface conditions of equation (53) are

$$\begin{aligned}u_{11}&=z_9+z_{10}y+z_{11}Cosh(My)+z_{12}Sinh(My)+k_{10}Cosh(2My)+k_{11}Sinh(2My)\\&+k_{12}ySinh(My)+k_{13}yCosh(My)+k_{14}y^2Cosh(My)+k_{15}y^2Sinh(My)+k_{16}y^4\\&+k_{17}y^3+k_{18}y^2\end{aligned} \qquad (56)$$

$$\begin{aligned}u_{21}&=z_{13}+z_{14}y+z_{15}Cosh(p_1y)+z_{16}Sinh(p_1y)+A_2\big(k_{28}Cosh(2p_1y)+k_{29}Sinh(2p_1y)\\&+k_{30}ySinh(p_1y)+k_{31}yCosh(p_1y)+k_{32}y^2Cosh(p_1y)+k_{33}y^2Sinh(p_1y)+k_{34}y^4\\&+k_{35}y^3+k_{36}y^2\big)\end{aligned} \qquad (57)$$

Using velocities given by equations (54)-(57), the expressions for energy balance equations (35) and (36) becomes

Region-I

$$\begin{aligned}\theta_1=\frac{1}{\Lambda}\Big(&-48-\varepsilon\big(3M^2k_{10}Cosh(2My)+3M^2k_{11}Sinh(2My)+2Mk_{12}Cosh(My)+\\&2Mk_{13}Sinh(My)+k_{14}\big(4MySinh(My)+2Cosh(My)\big)+k_{15}\big(4MyCosh(My)+\\&2Sinh(My)\big)+k_{16}\big(12y^2+M^2y^4\big)+k_{17}\big(6y+M^2y^3\big)+k_{18}\big(2+M^2y^2\big)\\&+M^2\big(z_9+z_{10}y\big)\big)+M^2E+M^2\big(z_1+z_2y\big)\Big)\end{aligned} \qquad (58)$$

Region-II

$$\theta_2 = \frac{1}{\Lambda bn}\Big(-48-\varepsilon\Big(A_2\left(3p_1^2k_{28}Cosh\left(2p_1y\right)+3p_1^2k_{29}Sinh\left(2p_1y\right)+2p_1k_{30}Cosh\left(p_1y\right)+ 2p_1k_{31}Sinh\left(p_1y\right)+k_{32}\left(4p_1ySinh\left(p_1y\right)+2Cosh\left(p_1y\right)\right)+k_{33}\left(4p_1yCosh\left(p_1y\right)+ 2Sinh\left(p_1y\right)\right)+k_{34}\left(12y^2+p_1^2y^4\right)+k_{35}\left(6y+p_1^2y^3\right)+k_{36}\left(2+p_1^2y^2\right)\right) + p_1^2\left(z_{13}+z_{14}y\right)\Big)+M^2E\sigma r\,Er+p_1^2\left(z_5+z_6y\right)\Big) \tag{59}$$

Solution with differential transformation method (DTM)

Now Differential Transformation Method has been applied to solving equations (27) and (28). Taking the differential transformation of equations (27) and (28) with respect to k, and following the process as given in Table 1 yields:

$$U\left(r+4\right)=\frac{1}{\left(r+1\right)\left(r+2\right)\left(r+3\right)\left(r+4\right)}\Big(M^2\left(r+1\right)\left(r+2\right)U\left(r+2\right) +\Lambda\,Br\sum_{s=0}^{r}\left(r-s+1\right)\left(s+1\right)U\left(r-s+1\right)U\left(s+1\right)+ \Lambda Br\left(M^2E^2\delta\left(r\right)+M^2\sum_{s=0}^{r}U\left(r-s\right)U\left(s\right)+2M^2EU\left(r\right)\right) \tag{60}$$

$$V\left(r+4\right)=\frac{1}{\left(r+1\right)\left(r+2\right)\left(r+3\right)\left(r+4\right)}\Big(M^2h^2\sigma r\,m\left(r+1\right)\left(r+2\right)V\left(r+2\right) +\Lambda\,Br\,nbkh^2\Big(mh^2\sum_{s=0}^{r}\left(r-s+1\right)\left(s+1\right)V\left(r-s+1\right)V\left(s+1\right) +M^2E^2\sigma r\,Er^2\delta\left(r\right)+M^2h^4\sigma r\,m^2\sum_{s=0}^{r}V\left(r-s\right)V\left(s\right) +2M^2Eh^2\sigma r\,ErmV\left(r\right)\Big) \tag{61}$$

The differential transform of the initial conditions are as follows

$$U\left(0\right)=c_1,\qquad U\left(1\right)=c_2,\qquad U\left(2\right)=\frac{c_3}{2},\qquad U\left(3\right)=\frac{c_4}{6},\quad V\left(0\right)=\frac{c_1}{mh^2},V\left(1\right)=\frac{c_2}{h},$$

$$V\left(2\right)=\left(c_3-M^2c_1+\frac{M^2\sigma r\,c_1}{nb}-A_{11}\right)\frac{nb}{2},\quad U\left(3\right)=\frac{\left(c_4-M^2c_2+\frac{A_{12}c_2}{h}\right)nbkh}{6} \tag{62}$$

Where $A_{11}=-48+M^2E+\frac{48}{nb}-\frac{M^2E\sigma r}{nb}$, $A_{12}=-\frac{M^2\sigma rhm}{nbk}$

Using the conditions as given in equation (62), one can evaluate the unknowns c_1, c_2, c_3, and c_4. By using the DTM and the transformed boundary conditions, above equations that finally leads to the solution of a system of algebraic equations.

A Nusselt number can be defined at each boundary, as follows:

$$Nu_1=\frac{2(h_1+h_2)}{R_T\left[T_2\left(h_2/2\right)-T_1(-h_1/2)\right]+(1-R_T)\Delta T}\left.\frac{dT_1}{dY}\right|_{Y=-h_1/2}$$

$$Nu_2=\frac{2(h_1+h_2)}{R_T\left[T_2\left(h_2/2\right)-T_1(-h_1/2)\right]+(1-R_T)\Delta T}\left.\frac{dT_2}{dY}\right|_{Y=h_2/2} \tag{63}$$

By employing equation (25), in equation (63) can be written as

$$Nu_1 = \frac{(1+h)}{R_T\left[\theta_2\left(1/4\right)-\theta_1(-1/4)\right]+(1-R_T)}\left.\frac{d\theta_1}{dy}\right|_{y=-1/4}$$

$$Nu_2 = \frac{(1+1/h)}{R_T\left[\theta_2\left(1/4\right)-\theta_1(-1/4)\right]+(1-R_T)}\left.\frac{d\theta_2}{dy}\right|_{y=1/4}. \quad (65)$$

IV. RESULTS AND DISCUSSION

In this section the fluid flow and heat transfer results for electrically conducting immiscible fluid flow in vertical channel are discussed in the presence of an applied magnetic field B_0 normal to gravity and applied electric field E_0 perpendicular to B_0 including the effects of both viscous and Ohmic dissipations. Robin boundary conditions for equal and unequal wall temperatures have been incorporated for the boundary conditions. The governing equations which are highly non-linear and coupled are solved by the well known perturbation method using the product of mixed convection parameter Λ and Brinkman number Br as perturbation parameter. The solutions obtained by perturbation method cannot be used for large values of perturbation parameters ε. However this condition on ε is relaxed by finding the solutions of the basic equations using DTM which is a semi analytical method. The electric field load parameter $E=0$ corresponds to short circuits configuration and $E \neq 0$ corresponds to open circuit configuration and the values of E may be taken as positive or negative depending on the polarity of E_0.

In the absence of viscous dissipation $(Br=0)$ and mixed convection parameter Λ, exact solutions can be obtained. The flow field for asymmetric heating and $Br=0$ is shown in figure 2. This figure indicates that there is a flow reversal near the cold wall for $\Lambda=1000$ and there is a symmetric profile for $\Lambda=0$ for both open and short circuits.

In the absence of mixed convection parameter Λ, plots of θ for equal and unequal Biot numbers are shown in figures 3a and 3b respectively for various values of Brinkman number. As the Brinkman number Br increases, temperature field is enhanced for both equal and unequal Biot number. The magnitude of enhancement at the cold wall is very large for unequal Biot numbers when compared with equal Biot numbers for all values of electric field load parameter E. Similar nature was also observed by Zanchini [26] for one fluid model for short circuits and in the absence of Hartman number.

The effect of Λ and ε on the flow field is shown in figures 4a and 4b for open circuit. It is seen that the velocity and temperature are increasing functions of ε for upward flow, velocity is a decreasing function of ε and temperature is an increasing function of ε for downward flow. The enhancement of flow for $\varepsilon>0$ implies that a greater energy generated by viscous dissipation yields a greater fluid temperature and as a consequence a stronger buoyancy force occurs. One can also reveal from figures 4a and 4b that the solution agree very well between perturbation method and differential transform method for $\varepsilon>0$ and differs slightly for $\varepsilon=2$ but becomes large for $\varepsilon=4$ for both buoyancy assisting $(\Lambda>0)$ and buoyancy opposing $(\Lambda<0)$ flows. Further there is a flow reversal at the cold wall for $\Lambda=500$ and at the hot wall for $\Lambda=-500$. The effects of ε and Λ on the flow was also the similar result observed for one fluid model for purely viscous fluid by Barletta [25] for isothermal wall conditions.

To understand the effects of Hartman number M and electric field load parameter E, the plots u and θ are drawn in figures 5a and 5b for equal Biot numbers. It is seen that for both open and short circuits the effects of M is to de-accelerate the flow. This is a classical Hartman result. The plots of u and θ for variations of viscosity ratio m is shown in figures 6a and 6b for variations of E. It is seen that as m increases velocity increases in region-I and decrease in region-II for both open and short circuits. Flow reversal is observed at the cold wall and the intensity of reversal flow increases for decreasing values of m. The slope at the interface suddenly drops for values of $m>1$ due to the condition imposed at the interface $(u_1 = mh^2u_2)$. Figure 6b shows that there is no effect of either m or E on the temperature field. The figures 7a and 7b refer to the influence of width ratio h on the velocity and temperature for open and short circuits. As h increases flow decreases in both the regions for open and short circuits. However the temperature filed is not affected by E. Further the flow reversal is also observed for variations of h at the cold wall and the intensity of reversal flow increases as h increases. One can observe from figures 6b and 7b that there is no drop of the slope at the interface on that temperature owing to the continuity of heat flux at the interface $\left(d\theta_1/dy = k\left(d\theta_2/dy\right)\right)$. Figures 8a and 8b exhibit the effect of thermal conductivity ratio k on the velocity and temperature fields. The effect of k is similar to the effect of h i.e. both the velocity and temperature decreases for increase in the value of k in both regions. Here also there is a flow reversal at the cold wall and the intensity of reversal flow increases with increase in k for all values of E .

The plots of u and θ are drawn in figure 9a and 9b for unequal Biot numbers with R_T (asymmetric wall heating condition) for both open and short circuits. It is seen from figure 9a that there is no flow reversal either at cold wall or at the hot wall as observed for equal Biot numbers for all values of E. However the effect of ε on u and θ for unequal Biot numbers remains the same for equal Biot numbers. The magnitude of temperature at the left wall for unequal Biot numbers is very large when compared to equal Biot numbers for all values of E.

Considering symmetric wall heat condition $(R_T = 0)$, the plots of u and θ are drawn and shown in figures 10a, b and 11a, b for equal and unequal Biot numbers for both open and short circuits. It is seen from these four figures that both u and $\Lambda\theta$ are increasing functions of ε for $E = 0, \pm 1$. Further the temperature profiles are symmetric for equal Biot numbers and the magnitude of temperature at the left wall is very large for unequal Biot numbers which was the similar nature observed for asymmetric wall heat conditions.

Figures 12a, b shows the plots of Nu_1 and Nu_2 for $\Lambda = 500, 100, -300$ versus $|\varepsilon|$ for both open and short circuits. These figures tells that Nu_1 is an increasing function of $|\varepsilon|$ while Nu_2 is decreasing function of $|\varepsilon|$ for both open and short circuits. Further the effects of ε on Nu_1 and Nu_2 is stronger for lower values of Λ for buoyancy assisting flow. In order to compare the present results with the earlier published work the values of viscosity ratio, width ratio and conductivity ratio taken as 1. The effects of Biot numbers on symmetric wall heat conditions and Nusselt numbers in the absence of Hartman number and electric field load parameter are the similar result observed by Zanchini [26] for one fluid model.

Tables 2-4 are the velocity and temperature solutions obtained by PM and DTM for symmetric and asymmetric wall heating conditions varying the perturbation parameter ε for equal and unequal Biot numbers. In Table 2, it is seen that in the absence of perturbation parameter, the PM and DTM solutions are equal for both the velocity and temperature fields. When the perturbation parameter ε is increased $(\varepsilon = 2)$, it is seen that the PM and DTM solutions do not agree. Similar nature is also observed in Table 3 and 4 for PM and DTM solutions. Table 2 and 3 are the solutions of velocity and temperature for asymmetric wall heating conditions for equal and unequal Biot numbers respectively. Table 2 and 3 also reveals that the percentage of error is large at the interface for velocity when compared with the error at the boundaries. Further the percentage of error between PM and DTM is large for unequal Biot numbers when compared with equal Biot numbers. Table 4 displays the solutions of symmetric wall heating conditions for equal Biot numbers. The percentage of error is less for symmetric wall heating conditions for equal Biot numbers when compared with asymmetric wall heat conditions.

Table 1: The operations for the one-dimensional differential transform method.

Original function	Transformed function
$y(x) = g(x) \pm h(x)$	$Y(k) = G(k) \pm H(k)$
$y(x) = \alpha\, g(x)$	$Y(k) = \alpha\, G(k)$
$y(x) = \dfrac{dg(x)}{dx}$	$Y(k) = (k+1)G(k+1)$
$y(x) = \dfrac{d^2 g(x)}{dx^2}$	$Y(k) = (k+1)(k+2)G(k+2)$

$y(x)=g(x)h(x)$	$Y(k)=\sum_{l=0}^{k}G(l)H(k-l)$
$y(x)=x^m$	$Y(k)=\delta(k-m)=\begin{cases}1, \text{ if } k=m\\ 0, \text{ if } k\neq m\end{cases}$

Table 2: Values of velocity and Temperature for $\Lambda=500$ and $R_T=1, M=4, E=-1$

y	$\varepsilon=0, Bi_1=Bi_2=10$			$\varepsilon=2, Bi_1=Bi_2=10$		
	PM	DTM	% error	PM	DTM	% error
-0.250	0.000000	0.000000	0.00%	0.000000	0.000000	0.00%
-0.150	0.276790	0.276790	0.00%	0.370820	0.385670	1.49%
-0.050	1.035680	1.035680	0.00%	1.184140	1.207620	2.35%
0.000	1.407780	1.407780	0.00%	1.565790	1.590860	2.51%
0.050	1.675850	1.675850	0.00%	1.830670	1.855350	2.47%
0.150	1.577240	1.577240	0.00%	1.684980	1.702290	1.73%
0.250	0.000000	0.000000	0.00%	0.000000	0.000000	0.00%
Temperature						
-0.250	-0.357140	-0.357140	0.00%	-0.351440	-0.350540	
-0.150	-0.214290	-0.214290	0.00%	-0.203310	-0.201580	0.17%
-0.050	-0.071430	-0.071430	0.00%	-0.056990	-0.054750	0.22%
0.000	0.000000	0.000000	0.00%	0.015240	0.017630	0.24%
0.050	0.071430	0.071430	0.00%	0.087010	0.089500	0.25%
0.150	0.214290	0.214290	0.00%	0.229940	0.232520	0.26%
0.250	0.357140	0.357140	0.00%	0.368830	0.370680	0.19%

Table 3: Values of velocity and Temperature for $\Lambda=500$ and $R_T=1, M=4, E=-1$

y	$\varepsilon=0, Bi_1=0.1, Bi_2=10$			$\varepsilon=2, Bi_1=0.1, Bi_2=10$		
	PM	DTM	% error	PM	DTM	% error
-0.250	0.000000	0.000000	0.00%	0.000000	0.000000	0.00%
-0.150	0.884080	0.884080	0.00%	1.060240	1.193770	13.35%
-0.050	1.334630	1.334630	0.00%	1.575460	1.757670	18.22%
0.000	1.407780	1.407780	0.00%	1.648480	1.830280	18.18%
0.050	1.376900	1.376900	0.00%	1.599440	1.767120	16.77%
0.150	0.969960	0.969960	0.00%	1.109010	1.212970	10.40%
0.250	0.000000	0.000000	0.00%	0.000000	0.000000	0.00%
Temperature						
-0.250	-0.023580	-0.023580	0.00%	0.005940	0.028420	2.25%
-0.150	-0.014150	-0.014150	0.00%	0.013640	0.034770	2.11%
-0.050	-0.004720	-0.004720	0.00%	0.019390	0.037730	1.83%
0.000	0.000000	0.000000	0.00%	0.022080	0.038840	1.68%
0.050	0.004720	0.004720	0.00%	0.024730	0.039840	1.51%
0.150	0.014150	0.014150	0.00%	0.029720	0.041100	1.14%
0.250	0.023580	0.023580	0.00%	0.032630	0.038930	0.63%

Table 4: Values of velocity and Temperature for $R_T=0, M=4, E=-1$

y	$\varepsilon=0, Bi_1=Bi_2=10$			$\varepsilon=2, Bi_1=Bi_2=10$		
	PM	DTM	% error	PM	DTM	% error
-0.250	0.000000	0.000000	0.00%	0.000000	0.000000	0.00%
-0.150	0.927020	0.927020	0.00%	0.978550	0.985610	0.71%
-0.050	1.355770	1.355770	0.00%	1.432300	1.442850	1.06%
0.000	1.407780	1.407780	0.00%	1.487380	1.498360	1.10%

0.050	1.355770	1.355770	0.00%	1.432300	1.442850	1.06%
0.150	0.927020	0.927020	0.00%	0.978550	0.985610	0.71%
0.250	0.000000	0.000000	0.00%	0.000000	0.000000	0.00%
Temperature						
-0.250	0.000000	0.000000	0.00%	2.325800	2.629610	30.38%
-0.150	0.000000	0.000000	0.00%	3.518370	3.993670	47.53%
-0.050	0.000000	0.000000	0.00%	3.704640	4.220740	51.61%
0.000	0.000000	0.000000	0.00%	3.712810	4.232370	51.96%
0.050	0.000000	0.000000	0.00%	3.704640	4.220740	51.61%
0.150	0.000000	0.000000	0.00%	3.518370	3.993670	47.53%
0.250	0.000000	0.000000	0.00%	2.325800	2.629610	30.38%

V. Conclusions

The problem of steady, laminar mixed convective flow in a vertical channel filled with electrically conducting immiscible fluid in the presence of viscous and Ohmic dissipation is analyzed using Robin boundary conditions. The governing equations were solved analytically using perturbation method valid for small values of perturbation parameter and by differential transform method valid for all values of governing parameters. The following conclusions were drawn.

[1] The flow at each position was an increasing function of ε for upward flow and decreasing function of ε for downward flow.
[2] The Hartman number suppresses the flow for symmetric and asymmetric wall heating conditions for all the governing parameters for both open and short circuits.
[3] Flow reversal was observed for asymmetric wall heating for equal Biot numbers and there is no flow reversal for unequal Biot numbers.
[4] The viscosity ratio increases the flow increases in region-I and decreases in region-II for equal Biot numbers. The width ratio and conductivity ratio suppress the flow in both the regions for equal Biot numbers.
[5] The Nusselt number at the cold wall was increasing function of $|\varepsilon|$ and decreasing function of $|\varepsilon|$ at the hot wall.
[6] The flow profiles for short circuit lie in between open circuit for positive and negative values of electric field load parameter.
[7] The percentage of error between PM and DTM agree very well for small values of perturbation parameter.
[8] Fixing equal values for viscosity, width and conductivity for fluids in both the regions and in the absence of Hartman number and electric field load parameter we get back the results of Zanchini [26] for one fluid model.

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REFERENCES

[1] Sparrow EM, Cess, RD (1961) Effect *of magnetic field on free convection heat transfer*. Int J Heat Mass Transfer 3: 267–274.
[2] Umavathi JC (1996) *A note on magneto convection in a vertical enclosure.* Int J Nonlinear Mechanics 31:371-376.
[3] Bhargava R, Kumar L, Takhar HS (2003) *Numerical solution of free convection MHD micropolar fluid flow between two parallel porous vertical plates*. Int J Engng Sci 41: 123-136.
[4] Hayat T, Wang Y, Hutter K (2004) *Hall effects on the unsteady hydromagnetic oscillatory flow if a second-grade fluid*. Int J Non-Linear Mech 39: 1027–1037.
[5] Umavathi JC, Liu IC, Prathap Kumar J, Pop I (2011) *Fully developed magnetoconvection flow in a vertical rectangular duct.* Heat Mass Transfer 47: 1-11.
[6] Charles ME, Redburger (1962) *On laminar two-phase flows in magnetohydrodynamics.* Can J Chem Eng 40: 70.
[7] Pacham BA, Shail R (1971) *Stratified laminar flow of two immiscible fluids.* Proc Camb Phil Soc 69: 443-448.
[8] Shail R (1973) *On laminar Two-phase flow in magnetohydromagnetics*. Int J Eng Sci 11:1103-1108.

[9] Postlethwaite AW, Sluyter MM (1978) *Magnetohydrodynamic heat transfer problems and overview.* Mech Eng 100: 29-32.
[10] Malashetty MS, Leela V (1992) *Magnetohydrodynamic heat transfer in two phase flow.* Int J Eng Sci 30: 371-377.
[11] Malashetty MS, Umavathi JC (1997) *Two phase magnethydrodynamic flow and heat transfer in an inclined channel.* Int J Multi Phase Flow 23: 545-560.
[12] Malashetty MS, Umavathi JC, Prathap Kumar J (2000) *Two fluid magnetoconvectionflow in an inclined channel.* Int J Transport phenomena 3: 73-84.
[13] Malashetty MS, Umavathi JC, Prathap Kumar J (2001) *Convective magnetohydrodynamic two fluid flows and heat transfer in an inclined channel.* Heat and Mass Transfer 37: 259-264.
[14] Malashetty MS, Umavathi JC, Prathap Kumar J (2006) *Magnetoconvection of two immiscible fluids in vertical enclosure.* Heat Mass Transfer 42: 977-993.
[15] Umavathi JC, Sridhar KSR (2009) *Hartman two fluid Poisuille-Couette flow in an inclined channel.* Int J Appl Mech Eng 14: 539-557.
[16] Umavathi JC, Liu IC, Prathap Kumar J (2010) *Magnetohydrodynamic Poiseuille couette flow and heat transfer in an inclined channel.* Journal of Mechanics 26.
[17] Umavathi JC, Prathap kumar J, Shekar M (2010) *Mixed convective flow of immiscible viscous fluids confined between a long vertical wavy wall and a parallel flat wall.* International journal of engineering science and technology 2:256-277.
[18] Prathap kumar J, Umavathi JC, Basavaraj M Biradar (2011) *Mixed convection of magneto hydrodynamic and viscous fluid in a vertical channel.* International Journal of Non-Linear Mechanics 46: 278–285.
[19] Prathap kumar J, Umavathi JC, Basavaraj M Biradar (2012) *Two-fluid mixed magnetoconvection flow in a vertical enclosure.* Journal of Applied Fluid Mechanics 5:11-21.
[20] Ingham DB, Keen DJ, Heggs PJ (1988) *Flows in a vertical channel with asymmetric wall temperatures and including situations where reverse flows occurs.* ASME J Heat Transfer 110: 910-917.
[21] Barletta A (1997) *Fully developed laminar forced convection in circular ducts for power law fluids with viscous dissipation.* Int J Heat and Mass Transfer 40:15-26.
[22] Zanchini E (1997) *Effect of viscous dissipation on the asymptotic behavior of laminar forced convection in circular tubes.* Int J Heat and Mass Transfer 40:169-178.
[23] Aung W, Worku G (1986) *Theory of fully developed combined convection including flow reversal.* ASME Journal of Heat Transfer 108:485–588.
[24] Cheng CH, Kou HS, Huang WH (1990) *Flow reversal and heat transfer of fully developed mixed convection in vertical channels.* Journal of Thermo physics and Heat Transfer 4: 375–383.
[25] Barletta A (1998) *Laminar mixed convection with viscous dissipation in a vertical channel.* Int J Heat and Mass Transfer 41:3501–3513.
[26] Zanchini E (1998) *Effect of viscous dissipation on mixed convection in a vertical channel with boundary conditions of the third kind.* Int J Heat and Mass Transfer 41:3949–3959.
[27] Stephan K (1959) *Warmenbergang a Druckabfall Bei nicht ausgebildeter laminarstromung in Rohren und in ebenen spalten.* Chemie Ingeniear Technik 31:773–778.
[28] Hwang CL, Fan LT (1964) *Finite difference analysis of forced convection heat transfer in entrance region of a flat rectangular duct.* Applied Scientific Research 13: 401–422.
[29] Siegal R, Sparrow EM (1959) *Simultaneous development of velocity and temperature distributions in a flat duct with uniform wall heating.* American Institute of Chemical Engineers Journal 5:73–75.
[30] Arikoglu A, Ozkol I (2006) *Solution of differential-difference equations by using differential transforms method.* Appl Math Comp 181: 153–62.
[31] Biazar J, Eslami M (2010) *Analytic solution for Telegraph equation by differential transform method .* Physics Letter A 374: 2904–2906.
[32] Zhou JK (1986) *Differential transformation and its applications for electrical circuits (in Chinese)* Huazhong University Press, Wuhan, China.
[33] Rashidi MM, Laraqi N, Sadri SM (2010) *A novel analytical solution of mixed convection about an inclined flat plate embedded in a porous medium using the DTM-Padé.* International Journal of Thermal Sciences 49: 2405–2412.
[34] Ganji DD, Rahimi M, Rahgoshay M, Jafari M (2011) *Analytical and numerical investigation of fin efficiency and temperature distribution of conductive, convective and radiative straight fins.* Heat Transfer – Asian Research 40: 233-245.

[35] Umavathi JC, Santhosh Veershetty (2012a) *Non-Darcy mixed convection in a vertical porous channel with boundary conditions of third kind.* Trans Porous Med 95:111–131.
[36] Umavathi JC, Santhosh Veershetty (2012b) *Mixed convection of a permeable fluid in a vertical channel with boundary conditions of a third kind.* Heat Transfer Asian Research 41.
[37] Umavathi JC, Jaweria Sultana (2011) *Mixed convection flow of conducting fluid in a vertical channel with boundary conditions of third kind in the presence of heat source/sink.* Int J Energy & Technology 3:1–10.

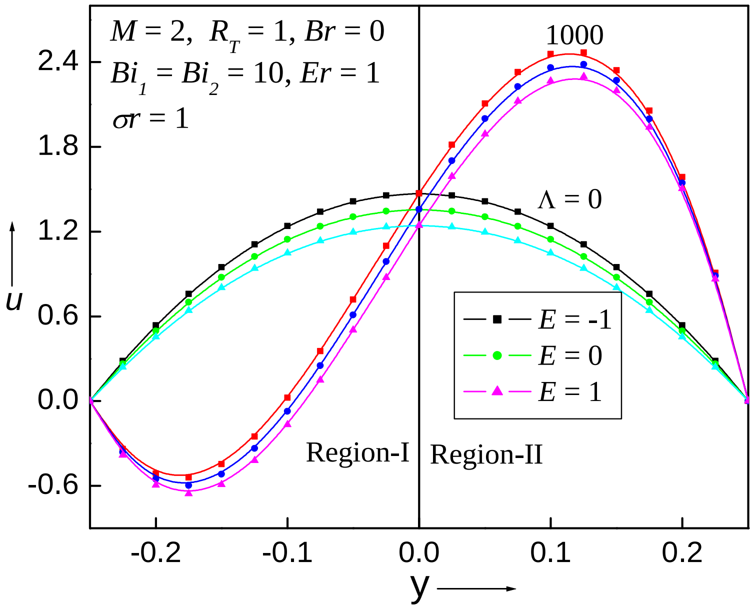

Figure 2. Plots of u versus y for different values of Λ

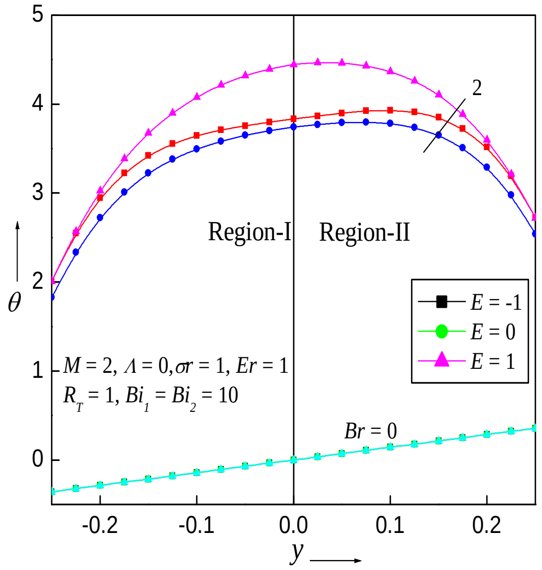

Figure 3a. Plots of θ versus y for different values of Br

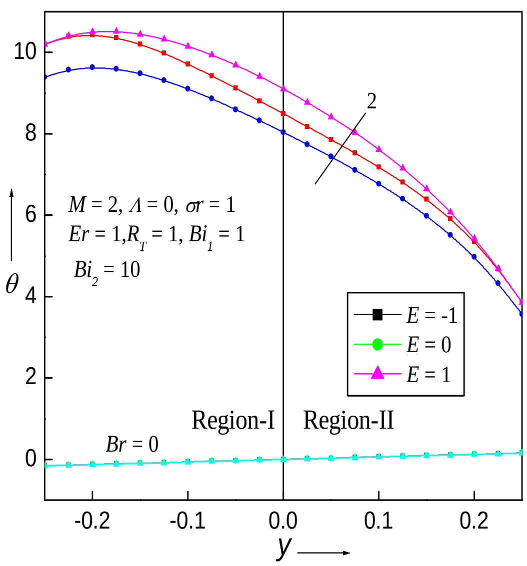

Figure 3b. Plots of θ versus y for different values of Br

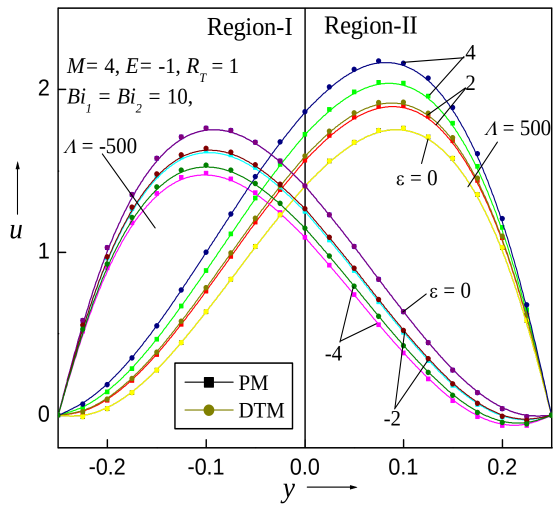

Figure 4a. Plots u versus y for different values of ε

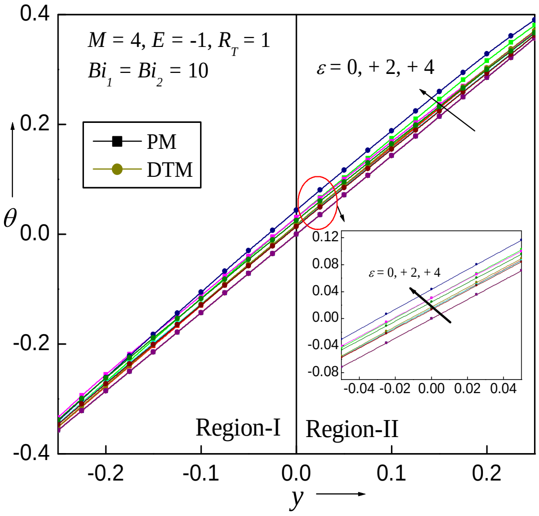

Figure 4b. Plots θ versus y for different values of ε

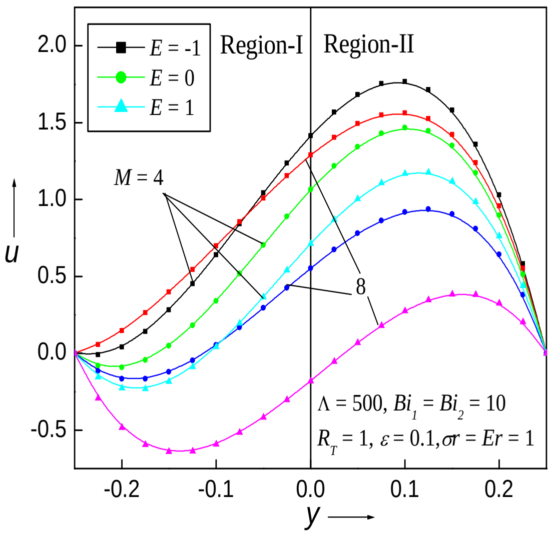

Figure 5a. Plots of u versus y for different values of M

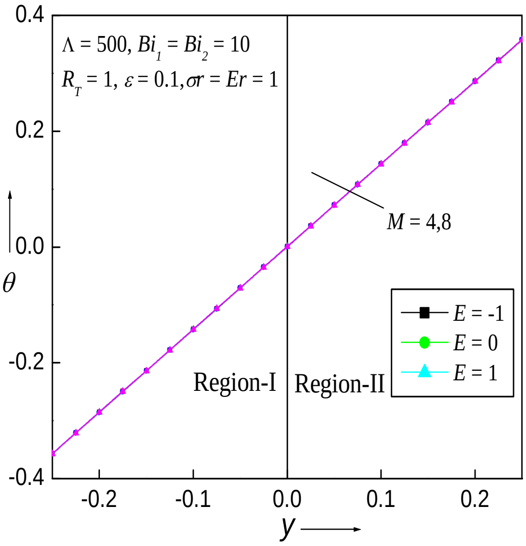

Figure 5b. Plots of θ versus y for different values of M

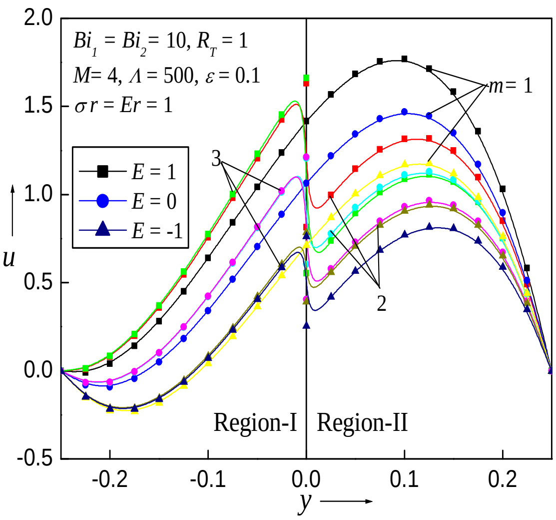

Figure 6a. Plots of *u* versus *y* for different values of *m*

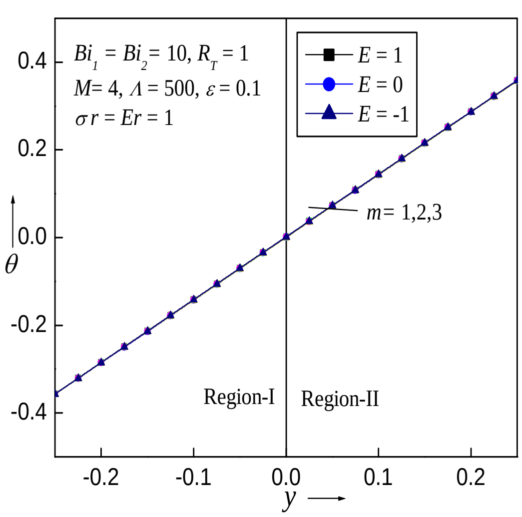

Figure 6b. Plots of θ versus *y* for different values of *m*

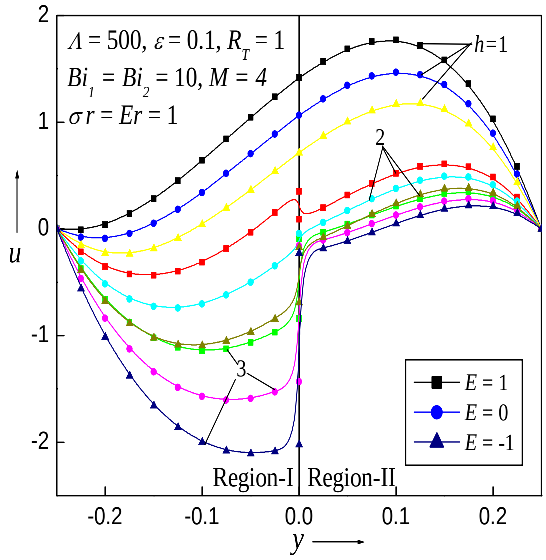

Figure 7a. Plots of *u* versus *y* for different values of *h*

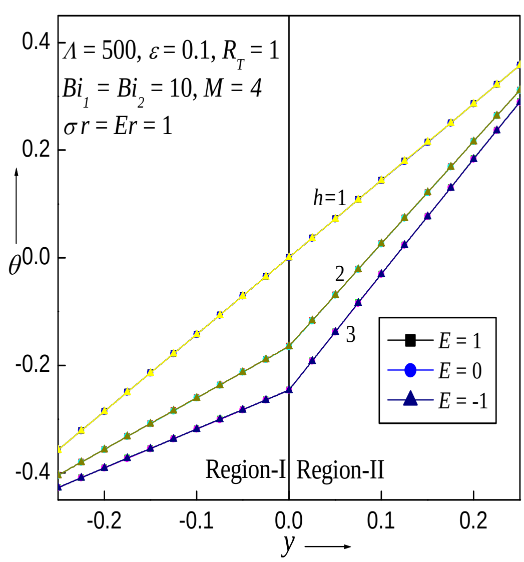

Figure 7b. Plots of θ versus *y* for different values of *h*

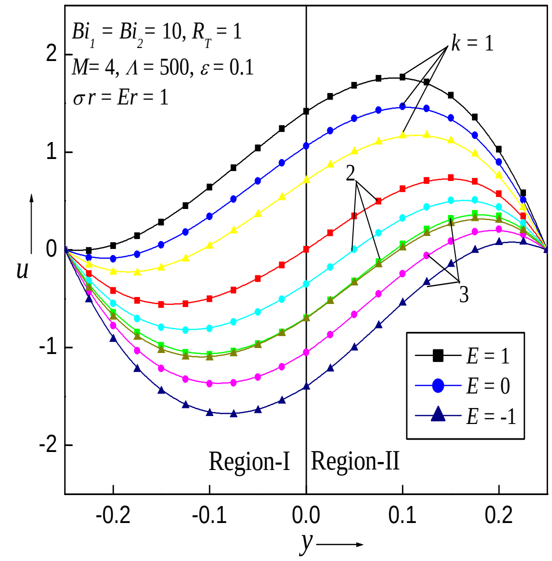

Figure 8a. Plots of u versus y for different values of k

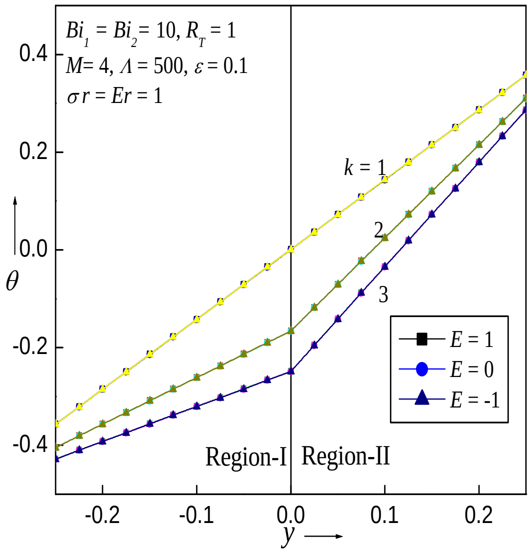

Figure 8b. Plots of θ versus y for different values of k

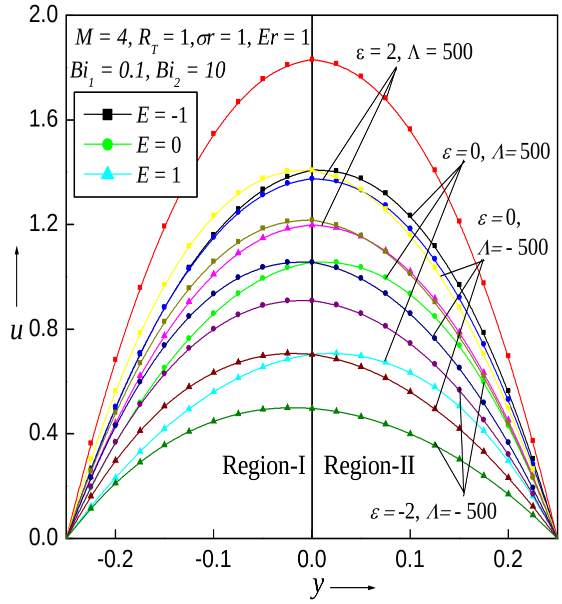

Figure 9a. Plots of u versus y for different values of ε

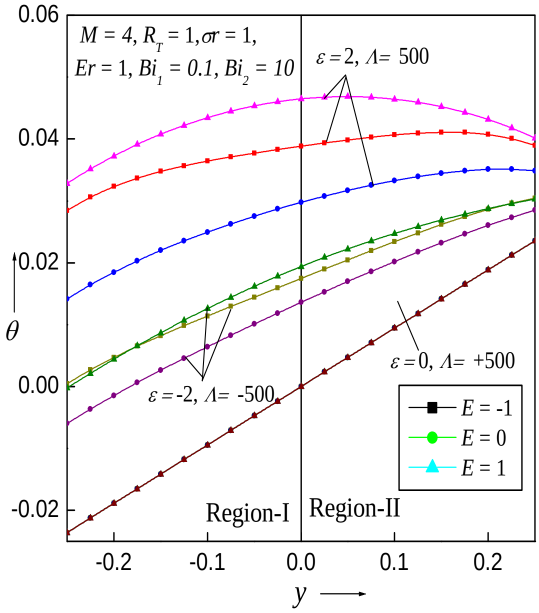

Figure 9b. Plots of θ versus y for different values of ε

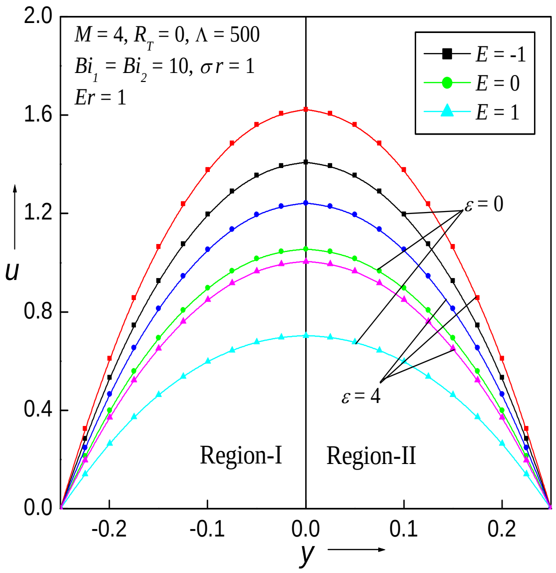

Figure 10a. Plots of u versus y for different values of ε

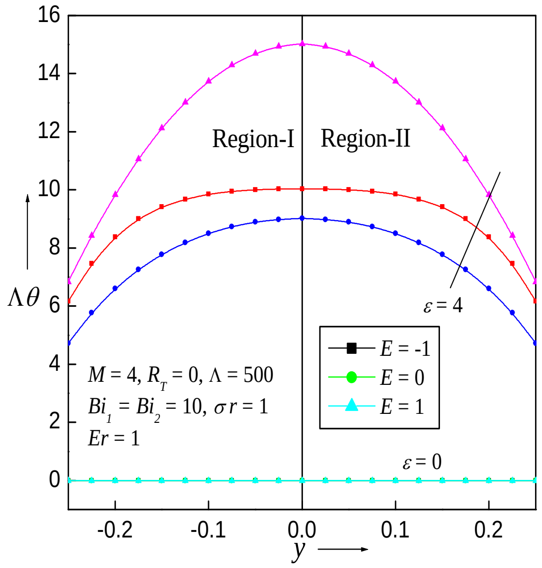

Figure 10b. Plots of θ versus y for different values of ε

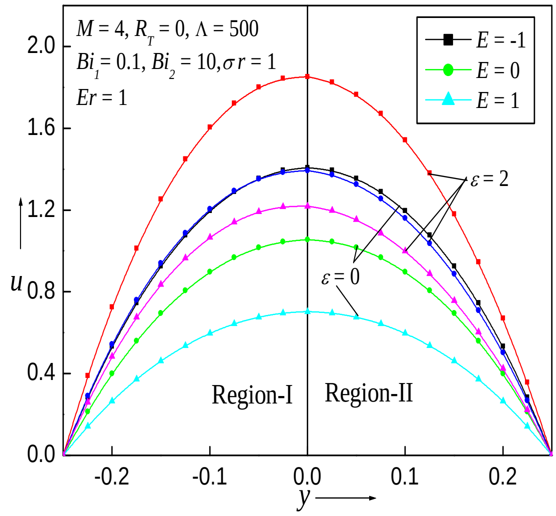

Figure 11a. Plots of u versus y for different values of ε

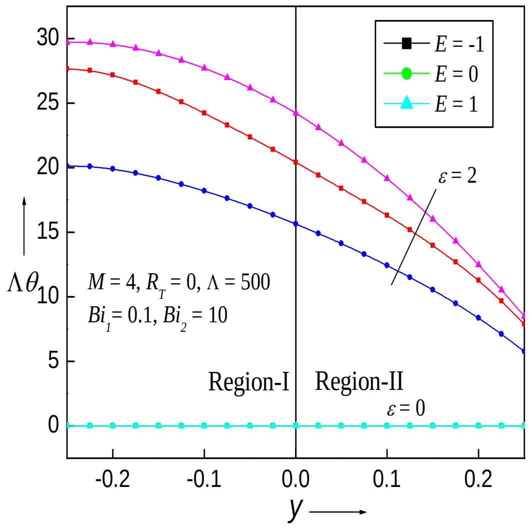

Figure 11b. Plots of θ versus y for different values of ε

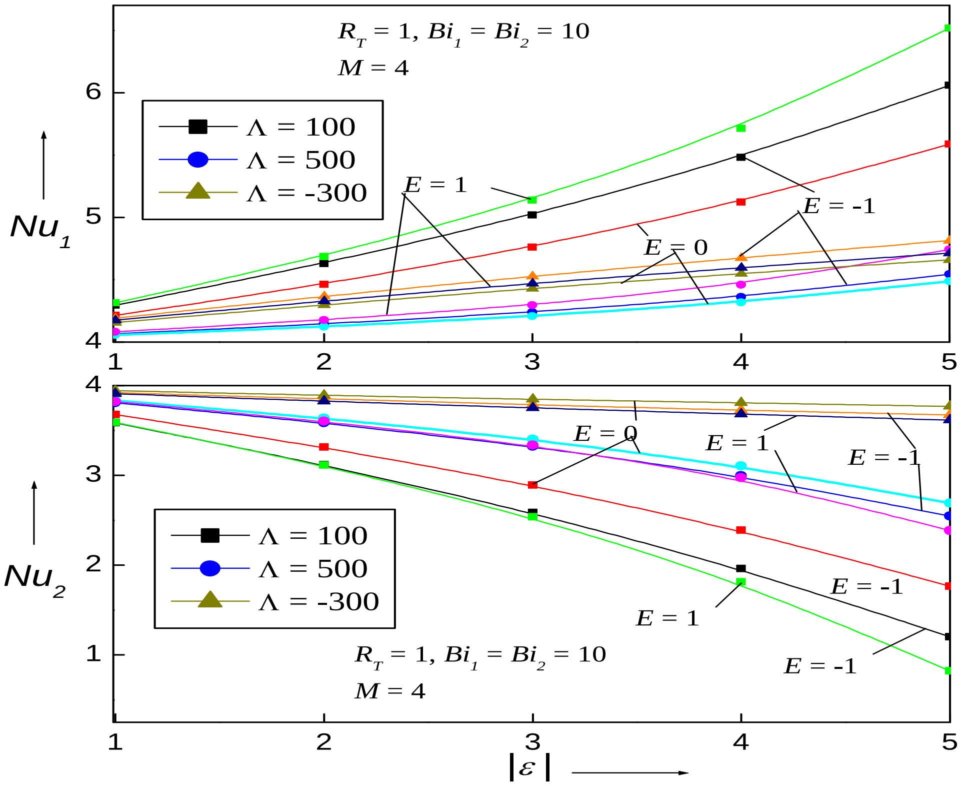

Figure 12. Plots of Nu_1 and Nu_2 versus $|\varepsilon$ for different values of Λ

NOMENCLATURE:

A constant used in equation (17)

Bi_1, Bi_2 Biot number $\left(q_i D_i / k_i\right)$

B_0 applied magnetic field

Br Brinkman number $\left(\mu_1 U_0^{(1)^2} / k_1 \Delta T\right)$

E_0 dimensional applied electric field

E dimensionless electric field load parameter $\left(E_0 / B_0 U_0^{(1)}\right)$

c_p specific heat at constant pressure

g acceleration due to gravity

Gr Grashof number $\left(g\,\beta_1 h_1^3 \Delta T / \upsilon_1^2\right)$

k ratio of thermal conductivities $\left(k_1 / k_2\right)$

h width ratio $\left(h_2 / h_1\right)$

M Hartman number $\left(\sigma_e B_0^2 D_1^2 / \mu_1\right)$

Nu_1, Nu_2 Nusselt numbers

p non-dimensional pressure gradient

P difference between pressure and hydrostatic pressure

q_1, q_2 external heat transfer coefficients
Re Reynolds number $\left(D_1 U_0^{(1)} / \nu_1\right)$
R_T temperature difference ratio
T temperature
T_{q_1}, T_{q_2} reference temperatures of the external fluid
T_0 reference temperature
U_i velocity component in the X -direction
$U_0^{(i)}$ reference velocity $\left(-AD_i^2 / 48\mu_i\right)$
u_i dimensionless velocity in the X -direction
X stream wise coordinate
x dimensionless stream wise coordinate
Y transverse coordinate
y dimensionless transverse coordinate

GREEK SYMBOLS
α_1, α_2 thermal diffusivities in region-I and region-II
β_1, β_2 thermal expansion coefficient in region-I and region-II
ΔT reference temperature difference $\left(T_{q_2} - T_{q_1}\right)$
ε perturbation parameter
θ_1, θ_2 dimensionaless temperatures in region-I and region-II
μ_1, μ_2 viscosities of the fluids in region-I and region-II
ν_1, ν_2 kinematic viscosities of the fluids in region-I and region-II
σ_{e1}, σ_{e2} electrical conductivities of the fluid in region-I and region-II
σr ratio of electrical conductivities $\left(\sigma_{e2} / \sigma_{e1}\right)$
ρ_1, ρ_2 density of fluids in region-I and region-II
Λ dimensionless mixed convection parameter $\left(Gr / \mathrm{Re}\right)$

SUBSCRIPTS
1 and 2 reference quantities for Region-I and II, respectively.

APPENDIX:

$$A_1 = M^2 h^2 \sigma_r m, \quad A_2 = h^2 bnk, \quad A_3 = mh^2, \quad A_4 = \sigma_r E_r^2 E^2 M^2, \quad A_5 = M^2 h^2 \sigma_r m^2,$$

$$A_6 = 2M^2 h^2 \sigma_r E E_r m, \quad A_7 = -\frac{M^2 h^2 \sigma_r m}{Bi_2}, \quad A_8 = -M^2 h^2 \sigma_r m,$$

$$BC_2 = \frac{-\Lambda bn R_T s}{2}(1 + \frac{4}{Bi_2}) - 48 + \sigma_r E_r E M^2, \quad A_9 = mh^2, \quad A_{10} = M^2 h^2 \sigma_r m,$$

$$A_{11} = -48 + M^2 E + \frac{48}{nb} - \frac{\sigma_r E_r E M^2}{nb}, \quad A_{12} = \frac{1}{nbkh}, \quad A_{13} = \frac{M^2 h \sigma_r m}{nbk}, \; cm_1 = Cosh[M/4]$$

$$sm_1 = Sinh[M/4], \; cm_2 = \cosh(\frac{p_1}{4}), \quad sm_2 = \sinh(\frac{p_1}{4}), \quad p_1 = \sqrt{A_1} \; p_2 = A_7 + \frac{A_8}{4},$$

$$p_3 = \frac{p_1^{\,3} sm_2}{Bi_2} + p_1^{\,2} cm_2 + A_7 p_1 sm_2 + A_8 cm_2, \quad p_4 = \frac{p_1^{\,3} cm_2}{Bi_2} + p_1^{\,2} sm_2 + A_7 p_1 cm_2 + A_8 sm_2,$$

$$p_5 = -\frac{p_1^{\,2}}{bn} + \frac{A_{10}}{bn}, \quad p_6 = \frac{A_{10}}{bn}, \quad p_7 = A_{12} p_1^{\,3} + A_{13} p_1, \quad p_8 = -(\frac{1}{Bi_1} + \frac{1}{4}), \quad p_9 = \frac{BC_1}{M^2},$$

$$p_{10} = -p_8 - \frac{1}{4}, \ p_{11} = -\frac{A_{13}}{M^2}, \ p_{12} = -\frac{p_7}{M^2}, \ p_{13} = p_{11} - h, \ p_{14} = p_{12} - p_1 h,$$

$$p_{15} = -\frac{p_{14}}{p_{13}}, \quad p_{16} = -\frac{M}{p_{13}}, \quad p_{17} = p_{10} p_{11} p_{15} + p_{10} p_{12}, \quad p_{18} = p_{10} p_{11} p_{16} - sm_2,$$

$$p_{19} = \frac{p_{15}}{4} + sm_2, \quad p_{20} = p_2 p_{15} + p_4, \quad p_{21} = -p_8 p_{11} p_{15} - p_8 p_{12}, \ p_{22} = -p_8 p_{11} p_{16},$$

$$p_{23} = M^2 p_8 p_{11} p_{15}, \quad p_{24} = M^2 p_8 p_{11} p_{16}, \quad p_{25} = -\frac{p_9}{cm_1}, \quad p_{26} = -\frac{p_{17}}{cm_1}, \quad p_{27} = -\frac{p_{18}}{cm_1},$$

$$p_{28} = p_{21} + p_{26}, \quad p_{29} = p_{22} + p_{27}, \quad p_{30} = p_9 + p_{25}, \quad p_{31} = \frac{A_{11} + M^2 p_9}{p_6}, \ p_{32} = -\frac{p_{23}}{p_6},$$

$$p_{33} = -\frac{p_{24}}{p_6}, \quad p_{35} = p_{32} + p_{19}, \quad p_{36} = p_{33} + \frac{p_{16}}{4}, \ p_{37} = p_{34} + cm_2, \quad p_{38} = A_8 p_{32} + p_{20},$$

$$p_{39} = A_8 + p_{33} + p_2 p_{16}, \ p_{40} = A_8 p_{34} + p_3, \quad p_{41} = A_8 p_{31} - BC_2, \quad p_{42} = p_{28} - A_9 p_{32},$$

$$p_{43} = p_{29} - A_9 p_{33}, \quad p_{44} = -A_9 p_{34} - A_9, \ p_{45} = p_{30} - A_9 p_{31}, \quad p_{46} = -\frac{p_{45}}{p_{43}}, \quad p_{47} = -\frac{p_{44}}{p_{43}},$$

$$p_{48} = -\frac{p_{42}}{p_{43}}, \ p_{49} = p_{31} + p_{36} p_{46}, \ p_{50} = p_{35} + p_{36} p_{48}, \ p_{51} = p_{37} + p_{36} p_{47},$$

$$p_{52} = p_{38} + p_{39} p_{48}, \quad p_{53} = p_{39} p_{47} + p_{40}, \quad p_{54} = p_{39} p_{46} + p_{41}, \ f_1 = 2M^2 E,$$

$$z_8 = -\frac{p_{49} p_{53} + p_{51} p_{54}}{p_{50} p_{53} - p_{51} p_{52}}, \quad z_7 = -\frac{p_{49}}{p_{51}} - \frac{p_{50}}{p_{51}} z_8, \ z_4 = p_{46} + p_{47} z_7 + p_{48} z_8,$$

$$z_5 = p_{31} + p_{32} z_8 + p_{33} z_4 + p_{34} z_7, \quad z_3 = p_{25} + p_{26} z_8 + p_{27} z_4, \ z_6 = p_{15} z_8 + p_{16} z_4,$$

$$z_2 = p_{11} z_6 + p_{12} z_8, \quad z_1 = p_9 - z_2 p_8,$$

$$k_1 = z_3^{\,2} M^2 + z_4^{\,2} M^2, \ k_2 = 2 z_3 z_4 M^2, \ k_3 = 2 z_2 z_4 M + 2M^2 z_1 z_3 + f_1 z_3, \ k_4 = 2 z_3 z_4 M^2,$$

$$k_5 = 2 z_3 z_2 M + 2 z_1 z_4 M^2 + f_1 z_4, \ k_6 = 2 z_2 z_4 M^2, \ k_7 = M^2 z_2^{\,2}, \ k_8 = 2 z_1 z_2 M^2 + f_1 z_2,$$

$$k_9 = z_2^{\,2} + M^2 E^2 + M^2 z_1^{\,2} + f_1 z_1, \ k_{10} = \frac{k_1}{12M^4}, \ k_{11} = \frac{k_2}{12M^4}, \quad k_{12} = \frac{k_3}{2M^3} - \frac{5k_6}{4M^4},$$

$$k_{13} = \frac{k_5}{2M^3} - \frac{5k_4}{4M^4}, \quad k_{14} = \frac{k_6}{4M^3}, \ k_{15} = \frac{k_4}{4M^3}, \ k_{16} = -\frac{k_7}{12M^2}, \ k_{17} = -\frac{k_8}{6M^2},$$

$$k_{18} = -(\frac{k_7}{M^4} + \frac{k_9}{2M^2}), \quad k_{19} = A_5 z_7^{\,2} + A_5 z_8^{\,2}, \ k_{20} = 2A_3 z_7 z_8 p_1^{\,2},$$

$$k_{21} = 2A_3 z_6 z_8 p_1 + 2A_5 z_5 z_7 + A_6 z_7, \ k_{22} = 2A_5 z_7 z_6, \ k_{23} = 2A_3 z_6 z_7 p_1 + 2A_5 z_5 z_8 + A_6 z_8,$$

$$k_{25} = A_5 z_6^{\,2}, \ k_{26} = 2A_5 z_5 z_6 + z_6 A_6, \quad k_{27} = A_3 z_6^{\,2} + A_4 + A_5 z_5^{\,2} + A_6 z_5, \ k_{28} = \frac{k_{19}}{12 p_1^{\,4}},$$

$$k_{29} = \frac{k_{20}}{12 p_1^{\,4}}, \ k_{30} = \frac{k_{21}}{2 p_1^{\,3}} - \frac{5k_{24}}{4 p_1^{\,4}}, \ k_{31} = \frac{k_{23}}{2 p_1^{\,3}} - \frac{5k_{22}}{4 p_1^{\,4}}, \quad k_{32} = \frac{k_{24}}{4 p_1^{\,3}}, \ k_{33} = \frac{k_{22}}{4 p_1^{\,3}},$$

$$k_{34}=-\frac{k_{25}}{12p_1^{\,2}},\quad k_{35}=-\frac{k_{26}}{6p_1^{\,2}},\quad k_{36}=-(\frac{k_{25}}{p_1^{\,4}}+\frac{k_{27}}{2p_1^{\,2}}),$$

$$f_2=k_{10}\cosh(\frac{M}{2})-k_{11}\sinh(\frac{M}{2})+\frac{k_{12}}{4}sm_1-\frac{k_{13}}{4}cm_1+\frac{k_{14}}{16}cm_1-\frac{k_{15}}{16}sm_1+\frac{k_{16}}{4^4}-\frac{k_{17}}{64}+\frac{k_{18}}{16},$$

$$f_3=A_2[k_{28}\cosh(\frac{p_1}{2})+k_{29}\sinh(\frac{p_1}{2})+\frac{k_{30}}{4}sm_2+\frac{k_{31}}{4}cm_2+\frac{k_{32}}{16}cm_2+\frac{k_{33}}{16}sm_2+\frac{k_{34}}{4^4}+\frac{k_{35}}{64}+\frac{k_{36}}{16}],$$

$$f_4=-\frac{3k_{16}}{4}-\frac{6k_{16}}{Bi_1}+\frac{3k_{17}}{2}+\frac{6k_{17}}{Bi_1}-2k_{18}+\frac{k_{16}M^2}{256}+\frac{k_{16}M^2}{16Bi_1}-\frac{k_{17}M^2}{64}-\frac{3k_{17}M^2}{16Bi_1}+\frac{k_{18}M^2}{16}+\frac{k_{18}M^2}{2Bi_1}$$
$$-2k_{14}cm_1-2k_{14}cm_1-2k_{12}Mcm_1+\frac{6k_{15}Mcm_1}{Bi_1}+\frac{2k_{13}M^2cm_1}{Bi_1}-\frac{k_{14}M^2cm_1}{Bi_1}-3k_{10}M^2\cosh(\frac{M}{2})$$
$$+\frac{6k_{11}M^2}{Bi_1}\cosh(\frac{M}{2})+2k_{15}sm_1+2k_{13}Msm_1-k_{14}Msm_1-\frac{6k_{14}M}{Bi_1}sm_1-\frac{2k_{12}M^2}{Bi_1}sm_1+\frac{k_{15}M^2}{Bi_1}sm_1$$
$$+3k_{11}M^2\sinh(\frac{M}{2}),\quad ff_5=\left(-\frac{M^2}{4}-\frac{M^2}{Bi_1}\right)$$

$$f_5=A_2A_8[\frac{k_{34}}{256}+\frac{k_{35}}{64}+\frac{k_{36}}{16}+\frac{k_{31}}{4}cm_2+\frac{k_{32}}{16}cm_2+k_{28}\cosh(\frac{p_1}{2})+(\frac{k_{30}}{4}+\frac{k_{33}}{16})sm_2+k_{29}\sinh(\frac{p_1}{2})]$$
$$+A_2A_7[\frac{k_{34}}{16}+\frac{3k_{35}}{16}+\frac{k_{36}}{2}+(k_{31}+\frac{k_{32}}{2}+\frac{k_{30}p_1}{4}+\frac{k_{33}p_1}{16})cm_2+2k_{29}p_1\cosh(\frac{p_1}{2})$$
$$+(k_{30}+\frac{k_{33}}{2}+\frac{k_{31}p_1}{4}+\frac{k_{32}p_1}{16})sm_2+2k_{28}p_1\sinh(\frac{p_1}{2})]+A_2[\frac{3k_{34}}{4}+\frac{3k_{35}}{2}+2k_{36}+(2k_{32}+2k_{30}p_1$$
$$+k_{33}p_1+\frac{k_{33}p_1^{\,2}}{16})cm_2+4k_{28}p_1^{\,2}\cosh(\frac{p_1}{2})+(2k_{33}+2k_{31}p_1+k_{32}p_1+\frac{k_{30}}{4}p_1^{\,2}+\frac{k_{33}}{16}p_1^{\,2})sm_2+$$
$$4k_{29}p_1^{\,2}\sinh(\frac{p_1}{2})]+\frac{1}{Bi_2}[A_2[6k_{34}+6k_{35}+(6k_{33}p_1+3k_{31}p_1^{\,2}+3\frac{k_{32}p_1^{\,2}}{2}+\frac{k_{30}p_1^{\,3}}{4}+\frac{k_{33}p_1^{\,3}}{16})cm_2+$$
$$8k_{29}p_1^{\,3}\cosh(\frac{p_1}{2})+(6k_{32}p_1+3k_{30}p_1^{\,2}+\frac{3k_{33}}{2}p_1^{\,2}+\frac{k_{31}p_1^{\,3}}{4}+\frac{k_{32}p_1^{\,3}}{16})sm_2+8k_{28}p_1^{\,3}\sinh(\frac{p_1}{2})]]$$

$$f_6=A_7+\frac{A_8}{4},\quad f_7=(A_7p_1+\frac{p_1^{\,3}}{Bi_2})cm_2+(A_8+p_1^{\,2})sm_2,\quad f_8=(A_8+p_1^{\,2})cm_2+(A_7p_1+\frac{p_1^{\,3}}{Bi_2})sm_2$$

$$f_9=k_{10}-A_2A_9k_{28},\quad f_{10}=k_{13}+2k_{11}M-A_2h(k_{31}+2k_{29}p_1),\quad f_{11}=2k_{14}+2k_{18}+2k_{12}M+3k_{10}M^2$$
$$\frac{A_{10}A_2k_{28}}{bn}-\frac{A_2}{bn}(2k_{32}+2k_{36}+2k_{30}p_1+4k_{28}p_1^{\,2}),\quad f_{12}=\frac{A_{10}}{bn},\quad f_{13}=\frac{A_{10}}{bn}+\frac{p_1^{\,2}}{bn},$$

$$f_{14}=6k_{17}+6k_{15}M+2k_{13}M^2+6k_{11}M^3-A_{13}A_2(k_{31}+2k_{29}p_1)-A_{12}A_2(6k_{35}+6k_{33}p_1+3k_{31}p_1^2$$
$$+8k_{29}p_1^3),\quad f_{15}=-A_{13}p_1-A_{12}p_1^3,\quad f_{16}=-\frac{f_4}{M^2},\ f_{17}=-\frac{ff_5}{M^2},\quad f_{18}=f_{17}-\frac{1}{4},\ f_{19}=f_2+f_{16}$$

$$f_{20}=f_9+f_{16},\quad f_{21}=f_{11}-M^2f_{16},\quad f_{22}=-M^2f_{17},\quad f_{23}=-\frac{cm_1}{f_{18}},\ f_{24}=\frac{sm_1}{f_{18}},\ f_{25}=-\frac{f_{19}}{f_{18}},$$

$$f_{26}=1+f_{17}f_{23},\quad f_{27}=f_{17}f_{25}+f_{20},\quad f_{28}=f_{10}+f_{25},\quad f_{29}=f_{24}+M,\quad f_{30}=f_{22}f_{25}+f_{21},$$

$$f_{31}=f_{22}f_{23},\quad f_{32}=f_{22}f_{24},\quad f_{33}=f_{14}-M^2f_{25},\ f_{34}=-M^2f_{23},\quad f_{35}=-M^2f_{24},\ f_{36}=-\frac{A_8}{4}+f_6$$

$$f_{37}=-A_8cm_2+f_8,\quad f_{38}=-A_8sm_2+f_7,\quad f_{39}=-A_8f_3+f_5,\quad f_{40}=A_9cm_2+A_9,\ f_{41}=A_9sm_2,$$

$$f_{42}=f_{17}f_{24},\quad f_{43}=A_9f_3+f_{27},\ f_{44}=f_{30}-f_{12}f_3,\ f_{45}=-\frac{f_{12}}{4},\quad f_{46}=-f_{12}cm_2+f_{13},$$

$$f_{47} = -f_{12} sm_2,\ \ f_{48} = -\frac{f_{37}}{f_{36}},\ \ f_{49} = -\frac{f_{38}}{f_{36}},\ \ f_{50} = -\frac{f_{39}}{f_{36}},\ \ f_{51} = \frac{A_9 f_{48}}{4} + f_{40},\ \ f_{52} = \frac{A_9 f_{49}}{4} + f_{41},$$

$$f_{53} = \frac{A_9 f_{50}}{4} + f_{43},\ \ f_{54} = f_{28} - hf_{50},\ \ f_{55} = -hf_{48},\ \ f_{56} = -hf_{49} - hp_1,\ \ f_{57} = f_{44} + f_{45} f_{50},$$

$$f_{58} = f_{46} + f_{45} f_{48},\ \ f_{59} = f_{47} + f_{45} f_{49},\ \ f_{60} = f_{33} - A_{13} f_{50},\ \ f_{61} = -A_{13} f_{48},\ \ f_{62} = f_{15} - A_{13} f_{49},$$

$$f_{63} = -\frac{f_{51}}{f_{26}},\ \ f_{64} = -\frac{f_{52}}{f_{26}},\ \ f_{65} = -\frac{f_{53}}{f_{26}},\ \ f_{66} = -\frac{f_{42}}{f_{26}},\ \ f_{67} = f_{54} + f_{23} f_{65},\ \ f_{68} = f_{55} + f_{23} f_{63},$$

$$f_{69} = f_{56} + f_{23} f_{64},\ \ f_{70} = f_{29} + f_{23} f_{66},\ \ f_{71} = f_{57} + f_{31} f_{65},\ \ f_{72} = f_{58} + f_{31} f_{63},$$

$$f_{73} = f_{59} + f_{31} f_{64},\ \ f_{74} = f_{32} + f_{31} f_{66},\ \ f_{75} = f_{60} + f_{34} f_{65},\ \ f_{76} = f_{61} + f_{34} f_{63},$$

$$f_{77} = f_{62} + f_{34} f_{64},\ \ f_{78} = f_{35} + f_{34} f_{66},\ \ f_{79} = -\frac{f_{67}}{f_{70}},\ \ f_{80} = -\frac{f_{68}}{f_{70}},\ \ f_{81} = -\frac{f_{69}}{f_{70}},$$

$$f_{82} = f_{71} + f_{74} f_{79},\ \ f_{83} = f_{72} + f_{74} f_{80},\ \ f_{84} = f_{73} + f_{74} f_{81},\ \ f_{85} = f_{75} + f_{78} f_{79},$$

$$f_{86} = f_{76} + f_{78} f_{80},\ \ f_{87} = f_{77} + f_{78} f_{81},\ \ z_{15} = -\frac{f_{84} z_{16}}{f_{83}} - \frac{f_{82}}{f_{83}},\ \ z_{16} = \frac{\left(f_{83} f_{85} - f_{82} f_{86} \right)}{f_{84} f_{86} - f_{83} f_{87}},$$

$$z_{12} = f_{79} + f_{80} z_{15} + f_{81} z_{16},\ \ z_{11} = f_{65} + f_{63} z_{15} + f_{64} z_{16} + f_{66} z_{12},\ \ z_{14} = f_{50} + f_{48} z_{15} + f_{49} z_{16},$$

$$z_{13} = -\left(f_3 + cm_2 z_{15} + sm_2 z_{16} + \frac{z_{14}}{4} \right),\ \ z_{10} = f_{25} + f_{23} z_{11} + f_{24} z_{12},\ \ z_9 = f_{16} + f_{17} z_{10}$$